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Density modulo 1 of Dilations of Sublacunary Sequences

MICHAEL D. BOSHERNITZAN*

Department of Mathematics, Rice University, Houston, Texas 77251

Let $\{r_i\}$ be a sequence of positive reals such that $r_{i+1}/r_i \to 1$ and $\limsup r_i = \infty$. Then the set $E = \{\alpha \in \mathbb{R} \mid \{\alpha r_i\}$ is not dense mod $1\}$ is of Hausdorff dimension 0. This result complements the following result of de Mathan and Pollington: If $\{r_i\}$ is lacunary, then the Hausdorff dimension of the set E is 1. © 1994 Academic Press, Inc.

NOTATION AND DEFINITIONS

Denote by $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ the sets of real numbers, integers, and positive integers, respectively. Denote by $\mathbb{I}$ the unit closed interval $[0, 1]$. A set $A \subset \mathbb{R}$ is said to be *dense modulo* 1 if the set

$$A + \mathbb{Z} \stackrel{\text{def}}{=} \{a + z \mid a \in A,\ z \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. For $\varepsilon > 0$, a set $A \subseteq \mathbb{R}$ is said to be *ε-dense modulo* 1 if the set $A + \mathbb{Z}$ intersects every open subinterval of $\mathbb{R}$ of length $\geqslant \varepsilon$.

A sequence $\mathbf{r} = \{r_i\}_{i \geqslant 1}$ of real numbers is said to be *dense modulo* 1 (*ε-dense modulo* 1) if the set $\{r_i \mid i \geqslant 1\}$ of its elements has the same property.

A sequence $\mathbf{r} = \{r_i\}$ of positive real numbers is said to be *lacunary* if

$$\liminf_{i \to \infty} \frac{r_{i+1}}{r_i} > 1.$$

We write $\dim_H E$ for the Hausdorff dimension of the set E.

1. MAIN RESULTS

The following result (which answered a question of Erdős [Er] was proved independently by de Mathan [M1], [M2] and Pollington [P].

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THEOREM 1.1. *Let* $\mathbf{r} = \{r_i\}$ *be a lacunary sequence of positive real numbers and define the set*

$$E = E(\mathbf{r}) = \{\alpha \in \mathbb{R} \mid \{\alpha r_i\} \text{ is not dense modulo } 1\}. \tag{1.2}$$

Then $\dim_{\mathrm{H}}(E(\mathbf{r}) \cap J) = 1$ *for every interval* $J \subset \mathbb{R}$.

It turns out that the situation is quite different if $r_{i+1}/r_i \to 1$. Our central result is

THEOREM 1.3. *Let* $\mathbf{r} = \{r_i\}$ *be an unbounded sequence of positive real numbers such that*

$$\lim_{i \to \infty} \frac{r_{i+1}}{r_i} = 1. \tag{1.4}$$

Then $\dim_{\mathrm{H}}(E(\mathbf{r})) = 0$ *(see* (1.2)*).*

We also show that Theorem 1.1 is in a sense best possible by proving the following

THEOREM 1.5. *Let* $\mathbf{r} = \{r_i\}$ *be an unbounded sequence of positive real numbers such that*

$$\frac{r_{i+1}}{r_i} < C, \qquad i \geqslant 1, \tag{1.6}$$

for some constant C. Then the set E is a countable union of sets of Hausdorff dimension < 1.

The proofs of Theorems 1.3, 1.5 are given in Sections 4 and 5. Note that the following is well known (cf. [KN, Chap. 1, Sect. 4, Corollary 4.3]).

PROPOSITION 1.7. *For any unbounded sequence* $\mathbf{r} = \{r_i\}$, *the set* $E = E(\mathbf{r})$ *(see* (1.2)*) has Lebesgue measure* 0.

For completeness, we present (in Section 6) a simple proof of Proposition 1.7 which is not based on Weyl's criterion for uniform distribution.

Some related results, both new and old, are reviewed in Sections 7 and 8.

2. More Definitions

For a set $X \subseteq \mathbb{R}$ and a closed interval $J = [a, b]$ in $\mathbb{R}$, denote

$$\Lambda(X, J) = \left\{ \frac{x-a}{b-a} \,\middle|\, x \in X \cap J \right\} \subseteq \mathbb{I} = [0, 1]. \tag{2.1}$$

Thus $A(X, J)$ is the image of the set $X \cap J$ under the linear transformation taking the interval J onto the unit interval $\mathbb{I}$. For two non-empty sets $A, B \in \mathbb{R}$, the Hausdorff distance $D(A, B) \in \mathbb{R}^+ \cup \{\infty\}$ is defined as follows:

$$D(A, B) = \sup_{a \in A} \inf_{b \in B} |b - a| + \sup_{b \in B} \inf_{a \in A} |b - a|. \tag{2.2}$$

Clearly, $D(A, B) = 0$ if and only if the closures of A and B coincide.

Denote by Φ the family of all closed subsets of $\mathbb{I}$. It is well known (and easy to see) that (Φ, D) is a compact metric space.

DEFINITION 2.3. The *family of limit sets* of a set $X \subseteq \mathbb{R}$, denoted by $\mathrm{FSL}(X)$, is the subfamily of Φ of all closed sets $Y \subseteq \mathbb{I}$ for which there exists a sequencev of intervals $\{J_i\}_{i \geq 1}$ in $\mathbb{R}$, with lengths $|J_i|$ approaching 0, such that

$$\lim_{\substack{D \\ i \to \infty}} A(X, J_i) = Y \tag{2.4}$$

(where $\lim_D$ stands for the limit in the Hausdorff pseudometric D).

Thus $\mathrm{FSL}(X)$ denotes the family of all limit sets obtained by choosing small intervals J and rescaling the set $X \cap J$.

DEFINITION 2.5. Let $X \subseteq \mathbb{R}$ and $k \geq 1$ be an integer. X is said to be *K-granular* is no set in $\mathrm{FLS}(X)$ contains more than k points. X is said to be *granular* if it is k-granular for some integer k.

EXAMPLES. A bounded $X \in \mathbb{R}$ is 1-granular if and only if X is finite. The sets $\{\sqrt{n} \mid n \in \mathbb{N}\}$, $\{1/n \mid n \in \mathbb{N}\}$ are not granular. $\{1/n! \mid n \in \mathbb{N}\}$ is 2-granular but not 1-granular.

PROPOSITION 2.6. *A set $X \subseteq \mathbb{R}$ is k-granular if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for each interval $J \subset \mathbb{R}$ of length $|J| < \delta$, the set $X \cap J$ can be covered by $\leq k$ intervals, each of length $< \varepsilon \cdot |J|$.*

The proof is straightforward.

Our proof of Theorem 1.3 (Section 4) is based on the fact that the exceptional set $E(\mathbf{r})$ (see (1.2)) is a countable union of granular sets (Proposition 4.1).

DEFINITION 2.7. A set $X \subseteq \mathbb{R}$ is said to be *perforated* if $\mathbb{I} \notin \mathrm{FSL}(X)$.

PROPOSITION 2.8. *A set $X \subseteq \mathbb{R}$ is perforated if and only if there exists $\varepsilon, \delta > 0$ such that, for every interval $J \subset \mathbb{R}$ of length $|J| < \delta$, there exists a subinterval $K \subseteq J$ of length $\varepsilon|J|$ such that $X \cap K = \varnothing$.*

The proof is straightforward.

Our proof of Theorem 1.5 (Section 5) is based on the fact that the exceptional set $E(\mathbf{r})$ is a countable union of perforated sets (Proposition 5.1).

3. Box Dimensions of Granular and Perforated Sets

Definition 3.1. For a bounded set $X \subset \mathbb{R}$, denote by $N_\delta(X)$ the smallest number of (say, closed) intervals of length at most δ which can cover X. The *(upper) box dimension* of X is defined as

$$\dim_B X = \limsup_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}.$$

It is easily seen (and well known; see [F, Chap. 3, Sect. 3.1]) that

$$\dim_H X \leqslant \dim_B X. \tag{3.2}$$

Proposition 3.3. *For a bounded granular set $X \subset \mathbb{R}$, $\dim_B X = 0$.*

Proof. X is k-granular for some $k \geqslant 1$ (Definition 2.5). In view of Proposition 2.6, it follows that, for every fixed $\varepsilon > 0$, $1 > \varepsilon > 0$, the inequality

$$N_{\varepsilon\delta}(X) \leqslant k N_\delta(X)$$

holds for all $\delta > 0$ small enough, and therefore

$$\limsup_{\delta \to 0} \frac{N_{\varepsilon\delta}(X)}{N_\delta(X)} \leqslant k.$$

The last inequality implies

$$\dim_B X = \limsup_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta} \leqslant \frac{\log k}{-\log \varepsilon},$$

and the claim of Proposition 3.3 follows since $\varepsilon > 0$ is arbitrary. ∎

Proposition 3.4. *For a bounded perforated set $X \subset \mathbb{R}$, $\dim_B X < 1$.*

Proof (similar to the previous one). Let $\varepsilon, \delta > 0$ be as in Proposition 2.8. Choose an integer $n > 1/\varepsilon$. It follows from Proposition 2.8 that, for every interval J of length $\leqslant d$, the set $X \cap J$ can be covered by $(n-1)$ intervals of length $\leqslant d/\mathrm{n}$. Thus

$$N_{d/n}(X) \leqslant (n-1)\, N_d(X).$$

and

$$\dim_B X = \limsup_{d \to 0} \frac{\log N_d(X)}{-\log d} \leqslant \frac{\log(n-1)}{\log n} < 1. \quad \blacksquare$$

4. Proof of Theorem 1.3

In view of (3.2) and Proposition 3.3, it suffices to prove the following

PROPOSITION 4.1. *Under the conditions of Theorem* 1.3, *the set E is a countable union of granular sets.*

For $x \in \mathbb{R}$, we denote by $\langle\langle x \rangle\rangle = x - [x]$ the fractional part of x. Proposition 4.1 follows from

PROPOSITION 4.2. *Let $K = (a, b) \subseteq \mathbb{I} = [0, 1]$, $0 \leqslant a < b \leqslant 1$. Under the conditions of Theorem* 1.3, *the set*

$$E(K) = E(K, \mathbf{r}) = \{\alpha \in \mathbb{R} \mid \langle\langle \alpha r_i \rangle\rangle \notin K, i \geqslant 1\}$$

is granular.

Proposition 4.2 implies Proposition 4.1 since $E = \bigcup E(K)$, the union being taken over the countable family of rational subintervals $K \subseteq \mathbb{I}$.

In the proof of Proposition 4.2, the following result is used.

PROPOSITION 4.3 [BP]. *For every $\varepsilon > 0$ there exists a positive integer $k = k(\varepsilon)$ such that for every set $X \subseteq \mathbb{I}$ of cardinality at least k the set*

$$n \cdot N = \{n \cdot x \mid x \in X\}$$

is ε-dense modulo 1 *for some integer $n \geqslant 1$.*

Remark. The study of the asymptotic behavior of $k(\varepsilon)$ as $\varepsilon \to 0$ was initiated by Berend and Peres [BP]. More precise estimates were recently obtained by Alon and Peres [AP]. The first result in this direction belongs to Glasner[G], who proved that for every infinite $X \subseteq \mathbb{I}$ and $\varepsilon > 0$ the set $n \cdot X$ is ε-dense modulo 1 for some $n \geqslant 1$. In this paper no quantitative estimates on $k(\varepsilon)$ are needed (except for $k(\varepsilon) < \infty$).

Proof of Proposition 4.2. Assume, to the contrary, that for some interval

$$K = (a, b) \subseteq \mathbb{I} = [0, 1], \qquad 0 \leqslant a < b \leqslant 1,$$

the set

$$E(K) = E(K, \mathbf{r}) = \{\alpha \in \mathbb{R} \mid \langle\!\langle \alpha r_i \rangle\!\rangle \notin K,\ i \geqslant 1\} \tag{4.4}$$

is not granular. Let $\varepsilon = b - a$. Since $E(K)$ is not granular, there exists a set $Y \in \mathrm{FLS}(E(K)) \subseteq \mathbb{I}$ containing at least $k = k(\varepsilon/2)$ points (see Definition 2.5 and Proposition 4.3). $Y \in \mathrm{FLS}(E(K))$ means (Definition 2.3) that there exists a sequence of intervals $\{J_i\}_{i \geqslant 1}$ in $\mathbb{R}$ with $|J_i|$ approaching 0 and

$$\lim_{\substack{D \\ i \to \infty}} \Lambda(E(K), J_i) = Y. \tag{4.5}$$

Put

$$J_i = (a_i, b_i) \tag{4.6}$$

and denote

$$d_i = |J_i| = b_i - a_i, \qquad c_i = \frac{a_i}{d_i}. \tag{4.7}$$

By the choice of J_i,

$$\lim_{i \to \infty} d_i = 0. \tag{4.8}$$

Since Y contains at least $k = k(\varepsilon/2)$ points, by Proposition 4.3 there exists an integer $m \geqslant 1$ such that the set

$$m \cdot Y = \{m \cdot y \mid y \in Y\} \tag{4.9}$$

is $\varepsilon/2$-dense modulo 1.

In view of (1.4) and (4.8), there exists a sequence $\mathbf{s} = \{s_i\}$ in the unbounded set $\{r_j \mid j \geqslant 1\}$ (i.e., $s_i \in \{r_j \mid j \geqslant 1\}$) such that

$$\lim_{i \to \infty} s_i \cdot \frac{d_i}{m} = 1. \tag{4.10}$$

Denote

$$U_i = J_i \cap E(K). \tag{4.11}$$

With the notation as in (4.7) and (2.1), we have

$$\Lambda(E(K), J_i) = \frac{U_i}{d_i} - c_i \overset{\text{def}}{=} \left\{ \frac{u}{d_i} - c_i \,\middle|\, u \in U_i \right\},$$

thus (4.5) can be rewritten in the form

$$\lim_{\substack{D \\ i \to \infty}} \left(\frac{U_i}{d_i} - c_i \right) = Y$$

whence

$$\lim_{\substack{D \\ i \to \infty}} f_i(U_i) = mY \tag{4.12}$$

where

$$f_i(x) \stackrel{\text{def}}{=} \frac{m}{d_i} x - mc_i \tag{4.13}$$

and

$$f_i(U_i) \stackrel{\text{def}}{=} \{f_i(u) \mid u \in U_i\}.$$

Observe that

$$\lim_{i \to \infty} D(f_i(U_i), g_i(U_i)) = 0 \tag{4.14}$$

where (see (4.6))

$$g_i(x) \stackrel{\text{def}}{=} s_i x - mc_i + \left(\frac{m}{d_i} - s_i\right) a_i \tag{4.15}$$

because

$$\lim_{i \to \infty} |f_i(x) - g_i(x)| = 0,$$

uniformly for $x \in U_i \subset J_i = (a_i, b_i)$.

Indeed,

$$|f_i(x) - g_i(x)| = \left|\left(\frac{m}{d_i} - s_i\right)(x - a_i)\right| \leqslant \left|\frac{m}{d_i} - s_i\right| |J_i| = |m - d_i s_i|$$
$$= m \cdot \left|1 - \frac{s_i d_i}{m}\right| \xrightarrow[i \to \infty]{} 0$$

in view of (4.10).

Combining (4.14) and (4.12), we obtain

$$\lim_{\substack{D \\ i \to \infty}} g_i(U_i) = m \cdot Y. \tag{4.16}$$

Since the set $m \cdot Y$ is $(\varepsilon/2)$-dense modulo 1, the last equality implies that $g_i(U_i)$ is ε-dense modulo 1, for all i large enough. Fix such an i, $i = n$. Since $g_n(U_n)$ is just a shift of $s_n \cdot U_n$ (see (4.15)), the set $s_n \cdot U_n$ is also ε-dense modulo 1. Since ε is the length of the interval K, there exists

$$u \in U_n \subset E(K)$$

(see (4.11) for the last inclusion) such that $us_n \in K$ modulo 1, a contradiction with the definition (4.4) of $E(K)$ (since $s_n \in \{r_j \mid j \geq 1\}$).

The proofs of Propositions 4.2 and (consequently) 4.1 are complete. ∎

It follows from Proposition 4.1 and 3.3 that (under the conditions of Theorem 1.3) the set $E = E(\mathbf{r})$ is a countable union of sets each having box dimension 0. Thus the *modified box dimension* of the set E is 0 (for the definition see [F, Chap. 3, Sect. 3.1]), and, in particular, $\dim_H E = 0$ (see [F, (3.20)]).

5. Proof of Theorem 1.5

The proof is similar to that of Theorem 1.3. In view of (3.2) and Proposition 3.4, it suffices to prove

PROPOSITION 5.1. *Under the conditions of Theorem* 1.3, *the set* E *is a countable union of perforated sets.*

The role of Proposition 4.2 is played by

PROPOSITION 5.2. *Let* $K = (a, b) \subseteq \mathbb{I} = [0, 1]$, $0 \leq a < b \leq 1$. *Under the conditions of Theorem* 1.5, *the set*

$$E(K) = E(K, \mathbf{r}) = \{\alpha \in \mathbb{R} \mid \langle\langle \alpha r_i \rangle\rangle \notin K,\ i \geq 1\} \tag{5.3}$$

is perforated.

Proposition 5.2 is an immediate corollary of the following

LEMMA 5.4. *Let* $\mathbf{r} = \{r_i\}$ *be a sequence satisfying the conditions of Theorem* 1.5. *Let* $L \subseteq \mathbb{R}$ *be a set which is not perforated. Then, for every* $\varepsilon > 0$, *there is* $r \in \{r_i \mid i \geq 1\}$ *such that the set* rL *is* ε*-dense modulo* 1.

Proof of Lemma 5.4. By Proposition 2.8, for every $\varepsilon > 0$, there exists an interval J of length $|J| < 1/r_1$ such that every subinterval $K \subset J$ of length $(\varepsilon/C)|J|$ will intersect L. In view of (1.6), there exists $r \in \{r_i \mid i \geq 1\}$ such that $1 < r\,|J| \leq C$. Thus the interval $P = rJ$ is of length > 1, and every subinterval $Q \subset P$ of length $(\varepsilon r/C)|J|$ intersects rL. Since $(\varepsilon r/C)|J| \leq \varepsilon$, the set rL is ε-dense modulo 1. ∎.

It follows from Propositions 5.2 and 3.4 that (under the conditions of Theorem 1.5) the set $E = E(\mathbf{r})$ is a countable union of sets each having box dimension < 1. This completes the proof of Theorem 1.5 (see (3.2)).

6. A Short Proof of Proposition 1.7

The standard proof is derived from the following propositions by passing to an appropriate subsequence.

Proposition 6.1. [KN, Chap. 1, Sect. 4, Cor. 4.3]. *Let* $\mathbf{s} = \{s_i\}_{i \geqslant 1}$ *be a sequence of reals such that*

$$\liminf (s_{i+1} - s_i)) > 0.$$

Then the set

$$U = U(\mathbf{s}) = \{\alpha \in \mathbb{R} \mid \{\alpha s_i\} \text{ is not uniformly distributed modulo } 1\}$$

is of Lebesgue measure 0.

In the special case when $\{s_i\}$ is a sequence of integers, Proposition 4.1 belongs to H. Weyl (cf. [KN, Chap. 1, Sect. 4, Th. 4.1]). The proof is based on Weyl's criterion [KN, Chap. 1] for uniform distribution modulo 1.

We present here a simpler proof of Proposition 1.7. The proof is similar in spirit to that of Theorems 1.3, 1.5. The role of Proposition 4.2 is played by

Lemma 6.2. *Let* $K = (a, b) \subseteq \mathbb{I} = [0, 1]$, $0 \leqslant a < b \leqslant 1$. *Under the conditions of Proposition* 1.7, *the set*

$$E(K) = E(K, \mathbf{r}) = \{\alpha \in \mathbb{R} \mid \langle\!\langle \alpha r_i \rangle\!\rangle \notin K,\ i \geqslant 1\}$$

is of Lebesgue measure 0.

Proof. The set $E(K)$ is clearly closed. Assume that $\mu(E(K)) > 0$, where μ denotes the Lebesgue measure on $\mathbb{R}$. Let $\varepsilon = b - a$. By the Lebesgue density theorem, one can choose, for every $\delta > 0$ small enough, an open interval $J = J(\delta) \subset \mathbb{R}$ of length δ such that

$$\mu(E(K) \cap J) > (1 - \varepsilon)\delta = (1 - \varepsilon)\, \mu(J). \tag{6.3}$$

Take $\delta = 1/|r_i|$ with $|r_i|$ large enough to ensure that there exists an interval J so that (6.3) holds. Then the set $r_i \cdot (E(K) \cap J) \subset \mathbb{R}$ has Lebesgue measure at least $1 - \varepsilon$ while it lies in the interval $U = r_i \cdot J$ of length 1. It follows that the set $r_i \cdot E(K)$ is ε-dense modulo 1 (because it is ε-dense in U). Since the length of the interval K is ε, there exists $x \in E(K)$ such that $r_i \cdot x \in K$ modulo 1, contradicting the definition of $E(K)$. ∎

7. Relatives of Theorem 1.3

We list some related results which can be proved by arguments similar to those applied in the proof of Theorems 1.3, 1.5.

For a real x, denote by $\|x\|$ the distance from x to the nearest integer. Thus $0 \leqslant \|x\| \leqslant \frac{1}{2}$, for all $x \in \mathbb{R}$.

THEOREM 7.1. *Let $\mathbf{r} = \{r_i\}$ be an increasing sequence of positive numbers such that*

$$\limsup_{i \to \infty} \frac{r_{i+1}}{r_i} < \infty. \tag{7.2}$$

Let $\{b_i\}$ be an arbitrary sequence of reals. Then the set

$$V = \{\alpha \in \mathbb{R} \mid \lim_{i \to \infty} \|\alpha \cdot r_i - b_i\| = 0\}$$

is countable (i.e., either infinitely countable or finite).

A special case (when $b_i = b$ for all i) of this theorem is proved in [Eg, Th. 16]. The theorem may be used for questions regarding density and uniform distribution of some *IP*-sequences [B]. Even though the proof in the general case is similar, we include it for completeness.

Remark 1. If instead of (7.2) one assumes that

$$\lim_{i \to \infty} \frac{r_{i+1}}{r_i} = \infty$$

then one can show that $\dim_{\mathrm{H}} V = 1$. (See [ET, Sect. 3, Th. 8] for a stronger result.)

Remark 2. The set V in Theorem 7.1 can be characterized explicitly, even without assuming (7.2), in the case when all $b_i = 0$ and $\mathbf{r} = \{r_i\}$ is the sequence of denominators q_i of the convergents p_i/q_i (continued fraction approximations) of some irrational number β (see [S], [L]). V is then countable if and only if β is badly approximable (i.e., has bounded quotients in its continued fraction expansion), in which case

$$V = \{n\alpha + m \mid n, m \in \mathbb{Z}\}.$$

Proof of Theorem 7.1. Put $\sup (r_{i+1}/r_i) = C$, $C > 1$. For an integer $n \geqslant 1$ and $\varepsilon > 0$, consider the set

$$V(n, \varepsilon) = \{\alpha \in \mathbb{R} \mid \|\alpha \cdot r_i - b_i\| < \varepsilon,\ i \geqslant n\}.$$

Since

$$V = \bigcap_{\varepsilon > 0} \left(\bigcup_{n \geqslant 1} V(n, \varepsilon) \right),$$

in order to prove the theorem, it suffices to verify that the set $V(n, \varepsilon)$ are countable provided $\varepsilon > 0$ is small enough.

We show that $V(n, \varepsilon)$ must be countable if $0 < \varepsilon < 1/4C$.

Assume to the contrary that $V(n, \varepsilon)$ is uncountable for some (fixed) $n \geqslant 1$ and $\varepsilon < 1/4C$.

Then each of the sets

$$V_i = r_i \cdot V(n, \varepsilon) - b_i, \qquad i \geqslant 1, \tag{7.3}$$

is uncountable. Therefore there exist points $x_n, y_n \in V_n$ such that

$$0 < |x_n - y_n| < \frac{1}{2C}. \tag{7.4}$$

Denote

$$x_i = \frac{r_i(x_n + b_n)}{r_n} - b_i; \qquad y_i = \frac{r_i(y_n + b_n)}{r_n} - b_i; \qquad i \geqslant 1. \tag{7.5}$$

One verifies easily that

$$x_i, y_i \in V_i, \qquad i \geqslant 1. \tag{7.6}$$

It follows from (7.4) and the inequalities

$$1 < \frac{|x_{i+1} - y_{i+1}|}{|x_i - y_i|} = \frac{r_{i+1}}{r_i} \leqslant C, \qquad i \geqslant 1,$$

that, for some $k > n$, the inequality

$$\frac{1}{2C} \leqslant |x_k - y_k| < \frac{1}{2}$$

takes place. The last inequality is incompatible with (7.6) due to the inequalities $k > n$, $C > 1$ and (see (7.3) and the definition of $V(n, \varepsilon)$),

$$\|z\| < \varepsilon < \frac{1}{4C}, \qquad z \in E_k.$$

The proof of Theorem 7.1 is complete. ∎

THEOREM 7.7. *Let* $\mathbf{r} = \{r_i\}$ *be as in Theorem* 1.3 *and* $\{b_i\}$ *an arbitrary sequence of reals. Then* $\dim_H E = 0$, *where*

$$E = \{\alpha \in \mathbb{R} \mid \{\alpha r_i - b_i\} \textit{ is not dense modulo } 1\}.$$

The proof is almost identical to that of Theorem 1.3.

Remark. Non-homogeneous versions (obviously stated) of Theorems 1.1, 1.5 also hold.

THEOREM 7.8. *Let* $\mathbf{t} = \{t_i\}$ *be an increasing sequence of real numbers such that*

$$\lim_{i \to \infty} (t_{i+1} - t_i) = 0. \tag{7.9}$$

Then, for every $a \neq 0$, $\dim_H B(a) = 0$, *where*

$$B(a) = \{b > 1 \mid \textit{the sequence } \{a \cdot b^{t_i}\} \textit{ is not dense modulo } 1\}.$$

The proof, being long, is not included. The argument is the same as that in the proof of Theorem 1.3.

Remark. If instead of (7.9) one has

$$\liminf_{i \to \infty} (t_{i+1} - t_i) > 0$$

then $\dim_H B(a) = 1$.

8. EXCEPTIONAL SETS FOR DENSITY AND UNIFORM DISTRIBUTION. REVIEW

It is known (see [Bos1], [AHK], [Bou]) that, for a "typical" sublacunary sequence $\mathbf{r} = \{r_i\}$, the set

$$U = U(\mathbf{r}) = \{\alpha \in \mathbb{R} \mid \{\alpha r_i\} \textit{ is not uniformly distributed modulo } 1\}$$

reduces to just $\{0\}$, while for a "typical" sublacunary sequence $\mathbf{r}$ of integers the set U reduces to the set of rational numbers. (In the above quoted papers "typical" has various probability meanings.) Denote as before

$$E = E(\mathbf{r}) = \{\alpha \in \mathbb{R} \mid \{\alpha r_i\} \textit{ is not dense modulo } 1\}.$$

Clearly, $E(\mathbf{r}) \subseteq U(\mathbf{r})$, for every sequence $\mathbf{r} = \{r_i\}$ of reals.

We can show the following

PROPOSITION 8.1. [Bos2]. *For a sequence* $\mathbf{r} = \{r_i\}$ *of reals such that*

$$\liminf_{i \to \infty} (r_{i+1} - r_i) > 0 \tag{8.2}$$

and

$$\liminf_{i \to \infty} \frac{r_i}{i} < \infty, \tag{8.3}$$

the set E is countable.

Remark. The result was recently proved independently by J. Rosenblatt.

The condition (8.3) on the linear growth of $\{r_i\}$ cannot be improved, i.e., replaced by faster growth at infinity. In particular, there exists an increasing sequence $\{r_i\}$ of *integers* satisfying (8.2) and such that

$$\lim_{i \to \infty} \frac{r_i}{i \log |\log i|} = 0,$$

for which the set $E(\mathbf{r})$ is uncountable (cf. Theorem 1.3).

Note that, even under conditions stronger than those in Theorem 8.1, ($\{r_i\}$ is an increasing sequence of integers satisfying (8.2) and such that $r_{i+1} - r_i < C < \infty$), the set $U(\mathbf{r})$ may be uncountable [ET, Theorem 11]. Nevertheless, it must satisfy $\dim_{\mathrm{H}}(U(\mathbf{r})) = 0$ [ET, Theorem 12].

On the other hand, one can have $\dim_{\mathrm{H}}(U(\mathbf{r})) = 1$ for sequences $\mathbf{r} = \{r_i\}$ satisfying the conditions of Theorem 1.3. We conclude by quoting the following result indicating the connection between the growth of $\mathbf{r} = \{r_i\}$ and $\dim_{\mathrm{H}}(U(\mathbf{r}))$.

PROPOSITION 8.4 [ET, Theorem 13]. *Assume that, for an increasing sequence* $\mathbf{r} = \{r_i\}$ *of integers, the inequalities*

$$r_i < Ci^{\rho} \qquad (i \geqslant 1).$$

hold for some $C > 0$ *and* $\rho > 1$. *Then*

$$\dim_H (U(\mathbf{r})) \leqslant 1 - 1/\rho.$$

Remark. The bound $1 - 1/\rho$ of this proposition can be obtained.

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